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Nonlinear dispersive equations for group analysis

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ABSTRACT

A class of dispersive equations is studied within the framework of group analysis of differential equations. The complete list of equivalence transformations is presented. It is shown that certain equations from the class admit nonclassical reductions. Potential and potential nonclassical symmetries are also considered.

KEY WORDS: Nonlinear dispersive equation, admissible transformations, Lie symmetries, Potential symmetries

1. INTRODUCTION

We consider the class of nonlinear dispersive equation (Rosenau, 2005).

$$E = u_t + \in \left(u^m\right)_x + \frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right]_x = 0 \qquad (1)$$

which is of interest in Mathematical Physics. Special cases of this class have been used to model successfully physical situations in a wide range of fields. For example, if a = 0 and b = n, we have generalization of KdV equation (Popovych, 2010; Rosenau, 1993)

$$E = u_t + \in \left(u^m\right)_x + \frac{1}{n}\left(u^n\right)_{xxx} = 0 \qquad (2)$$

and the equation that corresponds to the values m = 2, a = b = 1 describes the motion of a diluted suspension (Rosenau, 2006). Equations of the type (2) with values of the parameters m and n are denoted by K (m, n). For example, the properties of the equation K(2, 2) were examined in (Popovych, 2010). Further applications of the class (1) can be found in (Rosenau, 1994, 2000, 2005, 2006) and references therein.

Our goal in this chapter is to extend certain results of the recent work (Bruzon, 2012). In particular we give an enhanced Lie group classification for the class (1). The complete list of form-preserving point transformations is presented. We show the nonclassical reductions, potential symmetries and nonclassical potential symmetries.

Equivalence Transformations: We recall that an equivalence transformation of a PDEs, E(x, t, u) = 0, is an invertible transformation of the independent and dependent variables of the form

$$t' = Q(x,t,u), x' = P(x,t,u), u' = R(x,t,u) \qquad (3)$$

that maps every equation of the class into an equations of the same form, $E(x',t',u') = 0.$ A complete classifications of the transformations of the class (3) that connects equation (1) and

$$u'_{t'} + \in' \left(u'^{m'}\right)_{x'} + \frac{1}{b'}\left[u'^{a'}\left(u'^{b'}\right)_{x'x'}\right]_{x'} = 0 \qquad (4)$$

provides us the so called form preserving transformations (Kingston, 1998) (or admissible transformations (Ovsiannikov, 1959) of equation (1). Equivalence transformations can be regarded as a subset of such transformations.

In order to derive the desired equivalence group of transformations we need to consider two cases: (i) a + b -1 $\neq$ 0 and (ii) a + b -1 = 0.

case (i) If a + b -1 $\neq$ 0, we have $t' = \beta t + \gamma, x' = \alpha x + \delta, u' = \alpha^{-\frac{1}{a+b-1}}\beta^{\frac{3}{a+b-1}}u \qquad (5)$

where $\in' \beta^{-\frac{a+b-m}{a+b-1}} = \in \alpha^{\frac{a+b-3m+2}{a+b-1}}$

From the above relation we deduce that $\in \in' \succ 0$ or $\in = \in' = 0$ and α, β are nonnegative. Furthermore nonzero $\in$ *and* $\in' = 1$ and an equation with can be transformed into one with $\in' = -1.$ That is we can take without loss of generality $\in = \in' = \pm 1.$ Hence, for this last possibility, equation (1) admits a three parameter group of transformation.

$$t' = \alpha^{\frac{a+b-3m+2}{a+b-m}} t + \gamma, x' = \alpha x + \delta, u' = \alpha^{-\frac{2}{a+b-m}}u,$$
$$(a',b',m',\in = \pm 1) = (a,b,m,\in = \pm 1) \qquad (6)$$

While in the case $\in = \in' = 0$ equation (1) takes the form $u_t + \frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right] = 0 \qquad (7)$

which it admits the four-parameter equivalence group (5).

Now if $m = m' = 1$ we obtain the 4-parameter equivalence group

$$t' = \beta t + \gamma, x' = \alpha x + (\beta \in' - \alpha \in) t + \delta$$
$$u' = \alpha^{-\frac{1}{a+b-1}} \beta^{-\frac{3}{a+b-1}} u, \ (a', b' = (a, b)) \quad (8)$$

From this equivalence transformation we deduce that equations (1) and (4) are connected with $\in or \in'$ being zero. In the other words, in the case where m = 1 equation (1) can be mapped into equation (7).

case (ii) If a + b -1= 0,

In this case, which also implies that $a' + b' = 1$ we have $t' = \alpha^3 t + \gamma, x' = \alpha x + \delta, u' = \beta u$ where $\in \beta^{1-m} = \alpha^2 \in'$ (9)

As in the previous case we have $\in \in' \succ 0 \, or \in = \in' = 0$. Hence in the case $\in = \in' = \pm 1$ equation (1) admits a three parameter equivalence group $t' = \alpha^3 t + \gamma, x' = \alpha x + \delta, u' = \alpha^{\frac{2}{1-m}} u$ (10)

while in case $\in = \in' = 0$, it admits four parameter equivalence group (9). Finally if m = 1 which also implies that m' = 1, equation (1) admits four parameter equivalence group $t' = \alpha^3 t + \gamma, x' = \alpha x + (\alpha^3 \in' - \alpha \in) t\delta, u' = \beta u$. Clearly as in the previous case, if m = 1 equation (1) can be mapped simpler equation (7).

Theorem 1: Equation (1) admits

(1) a three parameter Lie group if a, b and m are arbitrary;

(2) a four parameter Lie group if (a) a, b are arbitrary and m = 0 and if (b) a = 0, b = 1 and m = 2;

(3) a five parameter Lie group of a = 0, b = -1/2 and m = 0;

(4) an infinite – dimensional Lie group if a = 0, b = 1 and m = 0.

Any other member of the class (1) is equivalent to the above five cases.

In this case m = 0 which is equation (7), if we set n = a + b – 1 and k = b -1, we obtain the a class of equations

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n \frac{\partial^2 u}{\partial x^2} - a u^{n-1} \left(\frac{\partial u}{\partial x} \right)^2 \right)$$

$u_t + \left[u^n u_{xx} + k u^{n-1} u_x^2 \right]_x = 0.$ The results on Lie symmetries presented in Theorem (1).

Invariant Solutions: We give optimal system which consists a list of inequivalent subalgebras for the three cases of Theorem (1) and we give some examples of reduced ODE's.

(1) Here we have three Lie symmetries $\Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = (a + b - 3m + 2) t \partial_t + (a + b - m) x \partial_x + 2u \partial_u$

that produce an optimal system which depend upon the values of the parameter a, b and m. We get four sub cases.

(i) a + b - 3m + 2 $\neq$ 0, a + b – m $\neq$ 0

$\langle \Gamma_3 \rangle, \langle \Gamma_1 + c\Gamma_2 \rangle, \langle \Gamma_2 \rangle$ where c = 0, $\pm$ 1. For each component of the optimal system we construct the corresponding similarity reduction that transforms (1) into an ODE. We obtain the following $\Gamma_2 \rangle : u = \phi(\omega), \omega = t$

$\langle \Gamma_1 + c\Gamma_2 \rangle : u = \phi(\omega), \omega = x - ct$

$\langle \Gamma_3 \rangle : u = t^{\frac{2}{a+b-3m+2}} \phi(\omega), \omega = x t^{-\frac{a+b-m}{a+b-3m+2}}$

The reduction that corresponds to the sub algebra $\langle \Gamma_1 + c\Gamma_2 \rangle$ leads to the equation $c\phi_\omega - \left[\phi^{a+b-2} \left(\phi \phi_{\omega\omega} + (b-1)\phi_\omega^2 \right) + \in \phi^m \right]_\omega = 0$

which provides travelling wave solutions for equation (1).

(ii) a + b – 3m + 2 $\neq$ 0, a + b – m = 0.

$\langle \Gamma_3 + \alpha\Gamma_2 \rangle, \langle \Gamma_1 + c\Gamma_2 \rangle, \langle \Gamma_2 \rangle$ where c = 0, ± 1 and α ϵ R. The sub algebra $\langle \Gamma_3 + \alpha\Gamma_2 \rangle$ produces the reduction

$u = x^{\frac{1}{(m-1)}} \phi(\omega), \omega = x^{-\frac{\alpha}{2(1-m)}} e^t$

$u = t^{\frac{1}{(m-1)}} \phi(\omega), \omega = t^{-\frac{\alpha}{2(1-m)}} e^x.$

(iii) a + b – 3m + 2 = 0 , a + b – m $\neq$ 0.

$\langle \Gamma_3 + \alpha\Gamma_1 \rangle, \langle \Gamma_1 + c\Gamma_2 \rangle, \langle \Gamma_2 \rangle$ where c = 0, ± 1 and $\alpha \in$ R. The sub algebra $\langle \Gamma_3 + \alpha\Gamma_1 \rangle$ produces the reduction $u = x^{\frac{1}{(m-1)}}\phi(\omega), \omega = x^{-\frac{\alpha}{2(1-m)}}e^{t}.$

(iv) a + b – 3m + 2 = 0, a + b – m = 0 $\Rightarrow$ m = 1, a + b = 1, $\langle \Gamma_3 + \alpha\Gamma_2 + \beta\Gamma_1 \rangle, \langle \Gamma_1 + \gamma\Gamma_2 \rangle, \langle \Gamma_2 \rangle$ where α, β, γ ϵ R.

The sub algebra $\langle \Gamma_3 + \alpha\Gamma_2 + \beta\Gamma_1 \rangle$ produces the reduction

$$u = e^{\frac{t}{\beta}}\phi(\omega), \omega = x - \frac{\alpha}{\beta}t, if\ \beta \neq 0,$$

$$u = e^{\frac{x}{a}}\phi(\omega), \omega = t\ if\ \ \beta = 0$$

In the case β ≠ 0 the sub algebra $\langle \Gamma_3 + \alpha\Gamma_2 + \beta\Gamma_1 \rangle$ leads to the equation $\alpha\phi_\omega - \phi - \beta\left[\phi_{\omega\omega} + (b-1)\phi^{-1}\phi_\omega^2 + \in \phi^m\right]_\omega = 0$

and in case β = 0, we obtain the solution $u = c_1 \exp\left[\frac{1}{\alpha^3}\left(\alpha^2 x - \left(\in \alpha^2 + b\right)t\right)\right]$

(2a) Here we have four Lie symmetries $\Gamma_1, \Gamma_2, \Gamma_3 (m = 0)$ and $\Gamma_4 = 3t\partial_t + x\partial_x$, which in addition to the sub bases 1(i)-1(iii) produce the reduction that corresponds to the sub algebra $\langle \Gamma_4 + c\Gamma_3 \rangle$ where c = 0 , $\pm$ 1. This sub algebra gives $u = t^{\frac{2c}{3+(a+b+2)c}}\phi(\omega), \omega = xt^{-\frac{1+(a+b)c}{3+(a+b+2)c}}$

if 3 + (a + b + 2) c $\neq$ 0,

$u = x^{-\frac{c}{c+1}}\phi(\omega), \omega = t$, if 3 + (a+b+2)c = 0

In the case 3 + (a + b + 2) c = 0, we obtain the solution $u = x^{-\frac{c}{c+1}}\left[\frac{3(a+b+2)(1+2b-a)t}{(a+b-1)^2} - c_1\right]^{-\frac{1}{a+b-1}}$

(2b) Here we have four Lie symmetries $\Gamma_1, \Gamma_2, \Gamma_3 (a = 0, b = 1, m = 2)$ and $\Gamma_4 = 2 \in t\partial_t + \partial_u$, which in addition to the sub case (1) (i) produce the reduction that corresponds to the sub algebra $\langle \Gamma_4 + c\Gamma_1 \rangle$ where c = 0, $\pm$ 1. We obtain $u = \frac{t}{c} + \phi(\omega), \omega = x - \frac{\in}{c}t^2,\ if\ c \neq 0$

$u = \frac{x}{2 \in t} + \phi(\omega), \omega = t\ if\ c = 0.$

In the case c $\neq$ 0 we obtain $\phi_{\omega\omega\omega} + 2 \in \phi\phi_\omega + \frac{1}{c} = 0.$ We integrate this equation and the integral has the form $\phi_{\omega\omega} + \in \phi^2 + \frac{\omega}{c} = 0$ where c_1 is the constant integration. Sub case of the above equation, is the first Painleve's transcendent with the form $\phi_{\omega\omega} + 6\phi^2 + \omega = 0.$ Finally, in the case c = 0, we obtain the solution $u = \frac{x + 2 \in c_1}{2 \in t}.$

(3) Here we have five Lie symmetries, $\Gamma_1, \Gamma_2, \Gamma_3 (a = 0, b = -1/2, m = 0), \Gamma_4 = 3t\partial_t + x\partial_x$ and $\Gamma_5 = x^2\partial_x + 4xu\partial_u$ which in addition to the sub cases 1(i) and 2(a) produce the reduction that corresponds to the sub algebra $\langle \Gamma_5 + \alpha\Gamma_2 + \beta\Gamma_4 \rangle$ where α, β ϵ R.

Potential Symmetries: In this case we consider the potential system, $v_x = u$

(11)

$v_t = - \in u^m - \frac{1}{b}\left[u_a\left(u^b\right)_{xx}\right]$, which it admits a Lie symmetries if and only if

$$\Gamma^{(1)}\left[v_x - u\right] = 0, \tag{12}$$

$$\Gamma^{(2)}\left[v_t + \in u^m + \frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right]\right] = 0 \tag{13}$$

for $v_x = u$ and $v_t = -\left[\in u^m + u^{a+b-1}u_{xx} + (b-1)u^{a+b-2}u_x^2\right]$ We recall that $\Gamma^{(1)}$, $\Gamma^{(2)}$ are the first and second extensions of the generator $\Gamma = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \zeta(t,x,u,v)\partial_v$

From the coefficients of $u_x u_{xx}, u_{xx}$ and u_x of (12) and from the coefficient $u_x u_{xx}$ of (13) we get that the coefficient τ is a function of t and the coefficient ξ and ζ are functions of t, x and v. From equation (12) we have that

$\eta = -\xi_v u^2 - (\xi_x - \zeta_v)u - \zeta_x$

After we have used the above results, we have that functions τ, ξ, η, ζ are satisfying the following determining system

$$(a+b+2)\xi_v u^2 - [\tau_t - (a+b+2)\xi_v + (a+b-1)\zeta_v]u$$
$$-(a+b-1)\zeta_x = 0 \qquad (14)$$
$$(b^2+ab+2b-a)\xi_v u^2 - (b-1)[\tau_t - (a=b+2)\xi_x]u$$
$$-(b-1)[(a+b-1)\zeta_v]u - (b-1)(a+b-2)\zeta_x = 0 \quad (15)$$

$$2(b+2)\xi_{vv}u^3 + [(4b+5)\xi_{vv} - (2b+1)\zeta_{vv}]u^2$$
$$+[(2b+1)\xi_{xx} - (4b-1)\zeta_{xv}]u - 2(b-1)\zeta_{xx} = 0 \quad (16)$$
$$\in (m-1)\xi_v u^{m+2} - \in [\tau_t - m\xi_x + (m-1)\zeta_v]u^{m+1}$$
$$-\in m\zeta_x u^m \xi_{vvv}u^{a+b+3} + 3(\xi_{xvv} - \zeta_{vvv})u^{a+b+3}$$
$$+3(\xi_{xxv} - \zeta_{xvv})u^{a+b+2} + (\xi_{xxx} - 3\zeta_{xxv})u^{a+b+1}$$
$$-\zeta_{xxx}u^{a+b} + \xi_t u^2 - \zeta_t u = 0 \qquad (17)$$

Equation (17) can break up into more equations in proportion the valves of the parameter a, b and m.

From the coefficient of u^2 in equation (14) we get two cases $\xi_v = 0$ or a + b + 2 = 0. After we have solved the determining system (14 – 17) we observe that for the case that $\xi_v = 0$ and a + b + 2 = 0 we do not find potential symmetries. The system (11) admits Lie symmetries which induce potential symmetries for the corresponding equation (1) in two cases.

(1) a + b + 2 = 0 and $\xi_v \neq 0$

(2) a + b + 2 ≠ 0 and $\xi_v = 0$.

We present only the potential symmetries when m ≠ 1. We use the equivalence transformations, equation (1) can be mapped into (7). We obtain that the following Lie symmetries of the system (11) induce potential symmetries for the corresponding equation (1)

(1) (a, b, m) = (0, -2, -1)

$\Gamma = v\partial_x - u^2\partial_u - 2\in t\partial_v$

(2) (a, b, m) = (3/2, -1/2, 3)

(i) ϵ > 0

$\Gamma_1 = \sqrt{2\in}uCos\left(\sqrt{2\in}v\partial_u\right) + Sin\sqrt{2\in}v\partial_v$

$\Gamma_2 = -\sqrt{2\in}uCos\left(\sqrt{2\in}v\partial_u\right) + Sin\sqrt{2\in}v\partial_v$

(ii) ϵ < 0

$\Gamma_1 = \sqrt{2|\in|}ue^{\sqrt{2|\in|}v}\partial_u + e^{\sqrt{2|\in|}v}\partial_v$

$\Gamma_2 = -\sqrt{2|\in|}ue^{-\sqrt{2|\in|}v}\partial_u + e^{-\sqrt{2|\in|}v}\partial_v$.

Further Potential Symmetries: Equation (1) can be written in other conserved forms when the parameter n, a and b satisfy certain relations. For example, if a ≠ b + 1, a ≠ b and m ≠ a – b, the auxiliary system takes the form

$$v_x = u^{b-a+1} \quad (17) \qquad v_t = (a-b-1)\begin{bmatrix} u^{2b-1}u_{xx} + \frac{1}{2}(a+b-2)u^{2b-2}u_x^2 \\ -\frac{\in m}{a-b-m}u^{b-q+m} \end{bmatrix}$$

for which Lie symmetries above induce potential symmetries of (1) in two cases. The first case is when (a, b, m) = (3, 1, 3) the system admits the following Lie symmetry which is a potential symmetry of (1).

$$\Gamma = v\partial_x - \partial_u - 6\in t\partial_v$$

The second case that the system (17) produces potential symmetries is when (a, b, m) = (3/2, -1/2, -1) and the symmetries have the form

(i) $\epsilon > 0$

$$\Gamma_1 = -\frac{2\in u}{3}e^{\sqrt{\frac{2\epsilon}{3}}v}\partial_u + \sqrt{\frac{-2\in}{3}}e^{-\sqrt{\frac{2\epsilon}{3}}v}\partial_v$$

$$\Gamma_2 = \frac{2\in u}{3}e^{-\sqrt{\frac{2\epsilon}{3}}v}\partial_u + \sqrt{\frac{2\in}{3}}e^{-\sqrt{\frac{2\epsilon}{3}}v}\partial_v$$

(ii) $\epsilon < 0$

$$\Gamma_1 = \frac{-2|\in|u}{3}Cos\left(\sqrt{\frac{2|\in|}{3}}v\right)\partial_u + \sqrt{\frac{2|\in|}{3}}Sin\left(\sqrt{\frac{2|\in|}{3}}v\right)\partial_v$$

$$\Gamma_2 = \frac{2|\in|u}{3}Sin\left(\sqrt{\frac{2|\in|}{3}}v\right)\partial_u + \sqrt{\frac{2|\in|}{3}}Cos\left(\sqrt{\frac{2|\in|}{3}}v\right)\partial_v$$

In case for which m = a – b , a ≠ b + 1 and a ≠ b equation (1) can be written as a system of two equations

$$v_x = u^{b-a+1},\ v_t = (a-b-1)\begin{bmatrix} u^{2b-1}u_{xx} + \frac{1}{2}(a+b-2)u^{2b-2}u_x^2 \\ +\in (a-b)\ln u \end{bmatrix}$$

If a = b and m ≠ 0, then equation (1) admits the conservation law

$$v_x = \ln u$$

$$v_t = -u^{2b-1}u_{xx} + \frac{1}{2}(1-2b)u^{2b-2}u_x^2 + \frac{\in m}{m-1}u^{m-1}$$

Lie symmetries of the above three systems lead only to Lie symmetries to (1).

2. RESULTS

From this paper we obtained potential symmetries form by substituted for a, b, m some values and discussed the cases (i) $\epsilon > 0$ and (ii) $\epsilon < 0$. Hence Lie symmetries of the all the above discussed three systems lead only to Lie symmetries to

$$E = u_t + \epsilon\left(u^m\right)_x + 1\left[u^a\left(u^b\right)_{xx}\right]_x = 0$$

3. CONCLUSION

The main goal of this paper was the investigation of symmetry properties for special classes of nonlinear evolution PDEs. Also we have given optimal system which consists a list of in equivalent sub algebras. Further potential symmetries were discussed; finally a peculiar result is obtained. We hope the results given in this paper will further enrich group analysis.

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